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Optimality of linearity with collusion and renegotiation☆

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HIGHLIGHTS

- We analyze a continuous-time Brownian agency model with constant absolute risk aversion utilities.
- N-agents determine the mean and variance of the returns.
- Our Brownian agency model features collusion and renegotiation.
- A theoretical justification for linear contracts is provided as in Holmstrom and Milgrom (1987).
- We prove that there exists a linear and stationary optimal compensation scheme.

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ABSTRACT

This study analyzes a continuous-time N-agent Brownian moral hazard model with constant absolute risk aversion (CARA) utilities, in which agents' actions jointly determine the mean and variance of the outcome process. In order to give a theoretical justification for the use of linear contracts, as in Holmstrom and Milgrom (1987), we consider a variant of its generalization given by Sung (1995), into which collusion and renegotiation possibilities among agents are incorporated. In this model, we prove that there exists a linear and stationary optimal compensation scheme which is also immune to collusion and renegotiation.

1. Introduction

We analyze contracting between a principal and a team of agents, where the outcome process is governed by a Brownian motion. Agents have CARA utilities and jointly determine the drift and diffusion rates. Each of them can observe the others' behavior and exploit any collusion and renegotiation opportunities at every instant via enforceable side-contracts contingent on effort levels and realized outcomes. We establish a theoretical justification for the use of linear contracts by proving that there are optimal stationary and linear sharing rules that are immune to collusion and renegotiation.[2] Thus, it is as if the agents were to choose the mean and variance only once and the principal were restricted to employ stationary and linear sharing rules.

Agents' ability to observe and verify others' actions and their knowledge of how each one of them affects the mean and variance as well as how these contribute to their costs bring about collusion and renegotiation concerns.[3] These, in turn, imply that agents' agreements have to be efficient. Alternatively, they have to solve

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2 Contacts generally have simpler forms (such as linear) compared to the ones predicted by the theory. As far as empirical evidence is concerned, Lafontaine (1992) reports that "franchise contracts generally involve the payment, from the franchisee to the franchisor, of a lump-sum franchise fee as well as a proportion of sales in royalties, with the latter usually constant over all sales levels". And, Slade (1996) notes that only linear contracts are used by the oil companies engaged in franchising in retail-gasoline markets in Vancouver.

3 This formulation is plausible when agents are better informed than the principal about the managerial details and interim outcomes of the project. This can occur when the principal does not have the necessary technical training (e.g., lacking the expertise to operate a nuclear power plant) to deal with the associated details which agents (well trained in nuclear physics and details about how to operate that power plant) are supposed to be fluent with in the first place. Or, when she is far away (e.g., in another country) from the agents (working in an overseas factory producing a technical product) and information technologies are not sufficient (possibly due to language barriers) so that the principal has to base her contract only on the final output, while agents working together (and speaking the same language) can observe and verify others' choices.

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a utilitarian bargaining in every date and state. The principal who cannot observe or verify agents' behavior only knows that the agents' bargaining (induced by her own offer) must result in an efficient outcome. Hence, the optimal contract she offers (i.e. individually rational sharing rules and control laws, drift and diffusion rates) must solve in every date and state agents' bargaining problem for some bargaining weights. Efficiency with CARA preferences delivers a useful aggregation result which we employ to establish that the principal can contract with the team as if she is contracting with a representative agent having CARA preferences, because we prove the following: given optimal control laws for the drift and diffusion rates and an optimal compensation for the team, agents' compensations obtained from the efficient distribution of team's compensations employing the stationary bargaining weights that are stated in our main result and the very same control laws, also solve agents' bargaining problem with their "real" date and state specific bargaining weights starting every date and state. Therefore, the fact that these particular bargaining weights are not necessarily agents' real bargaining weights (that the principal is not necessarily aware of) turns out not to be important. Due to the fact that, now, the principal is contracting with a representative agent having CARA preferences, her problem can be analyzed using techniques in Sung (1995) which enable us to establish that there is an optimal and stationary linear contract for the team. As linearity is preserved during the corresponding efficient redistribution of team's compensation to agents, our main result is established.[4]

Holmstrom and Milgrom (1987), the pioneer work displaying the optimality of linear contracts in a repeated agency setting with exponential utilities, considers a principal–agent pair where the agent determines the drift rate of a Brownian motion.[5] Schättler and Sung (1993) extends this setting by considering a larger class of stochastic processes. The key restriction in both models is that the agent is not allowed to control the variance of the outcome process. Sung (1995) extends Holmstrom and Milgrom (1987)'s Brownian model to the case where the agent can also control the diffusion rate of the Brownian motion. The resulting problem becomes similar to that in Holmstrom and Milgrom (1987) with an additional time-state independent constraint for which the linearity in outcome result holds. Koo et al. (2008), on the other hand, presents a continuous-time agency model under moral hazard with many agents.[6] They show that optimal contracts are also linear in all outcomes produced separately by each agent. For their linearity result, the formulation involving the simultaneous-move game played by agents is important to preserve stationary decision making environment. Meanwhile, our model does not feature separate production processes and our agents can perfectly observe each other and can engage in renegotiable side-contracting.

The paper is organized as follows. While Section 2 contains the model and the principal's problem, Section 3 presents the main result and its proof and Section 4 concludes.

2. Model and preliminaries

The principal and N agents interact over time interval $t \in [0, 1]$. At an instant t, agent $i \in N \equiv \{1, \ldots, N\}$ chooses an effort level $e_t^i \in E_i$, E_i a compact interval, and these choices are observable and verifiable by all the other agents, but not the principal. The probability space is given by $(\Omega, \mathcal{F}, P)$ where Ω is the space $C = C([0, 1])$ of all continuous functions on the interval $[0, 1]$ with values in $\Re$. So a particular event $w \in \Omega$ is of the form $w : [0, 1] \to \Re$. The effort choices $e : [0, 1] \to \times_{i\in N} E_i$, where $e_t = (e_t^i)_{i\in N}$, imply control laws μ and σ which are assumed to be $\mathcal{F}_t$-predictable mappings, $\mu : [0, 1]\times\Omega \to U$ and $\sigma : [0, 1]\times\Omega \to \mathbb{S}$, where U is a bounded open subset of $\Re$ and $\mathbb{S}$ is a compact subset of $\Re_{++}$. Controls μ and σ determine the instantaneous drift, μ_t, and the diffusion rates, σ_t, of a stochastic process, $\{X_t\}_t$, governed by a Brownian motion defined by $\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t$. Indeed, $\mu_t \equiv \mu(t, X)$ and $\sigma_t \equiv \sigma(t, X)$.[7]

The intermediate outcome X_t should be thought of as the total returns up to period $t \in [0, 1]$, and B_t is the standard Wiener process. The drift and diffusion rates and intermediate accumulated returns are neither observable nor verifiable by the principal. However, X_1, the level of accumulated returns at the end of the project, is observable and verifiable by the principal. At the beginning of the project, the principal and the agents agree upon a contract, i.e. salary rules $S = (S_i)_{i\in N}$ with $S_i : \Omega \to \Re$ for all $i \in N$ and control laws (μ, σ) with the restriction that salaries are payable at the end of the project according to the rules agreed upon at time 0 which depend only on X_1.[8]

Instantaneous time-state independent cost functions are given by $c_i(\mu_t, \sigma_t)$ where $c_i : U \times \mathbb{S} \to \Re$ is twice continuously differentiable, $i \in N$. c_i and $c_{i\mu}$ (derivative with respect to mean) are bounded, and both $c_{i\mu}$ and $c_{i\mu\mu}$ (second derivative with respect to mean) are strictly positive. The total costs incurred by agent $i \in N$ is given by $\int_0^1 c_i(\mu_t, \sigma_t)\mathrm{d}t$. In this setting there is an interaction effect on the two moments of the outcome process and on the costs of agents. Yet, it also handles the standard environment with two agents in which one agent determines only the mean and the other agent only the variance, and the interaction effect on the costs is assumed to be minimal.

All have CARA utilities where the coefficients of the principal and agent $i \in N$ are given by R and r_i, respectively. The reservation certainty equivalent agent $i \in N$ is given by W_{i0}. We assume that at each $t \in [0, 1]$, agents observe $h^t \equiv \{X_s, \mu_s, \sigma_s, (e_s^i)_{i\in N}\}_{s\le t}$. Agent i's expected utility at time t given $((S_i)_i, \mu, \sigma)$ (computed with the information at time t) is $E\left[-\exp\left\{-r_i W_i^S(X; \mu, \sigma)\right\}\middle| \mathcal{F}_t\right]$ where $W_i^S(X; \mu, \sigma) = \left(S_i(X) - \int_0^1 c_i(\mu_s, \sigma_s)\mathrm{d}s\right)$ is his net payoff at the end of the project.

At any t, h^t is observable and verifiable by all the agents but not the principal, and $(S_i)_{i\in N}$ is determined by the principal at

4 We thank an anonymous referee for pointing out that our analysis can be associated with bonus pools in investment banks. Bonus pools are allocated to divisions based on their performances. A division manager, who is given much flexibility, allocates the bonus to the employees. While some criticize nonuniform bonus allocations among employees on basis of fairness, our paper provides a justification: the bonus of an employee is determined through a utilitarian bargaining within the division, hence, depends on his relative bargaining power and risk preferences. The assumptions needed in this setting are: the employees cannot communicate with the shareholders; and the division manager knows the bargaining weights and risk aversion parameters of the employees, but the shareholders do not.

5 Lack of income effects with exponential utilities and time-state independent cost functions, imply that the optimal control the agent chooses is time-state independent. Stationarity of the environment implies that among all possible compensation schemes, an optimal one is stationary and linear in the final output.

6 Their model is a continuous-time counterpart of Holmstrom (1982) and an extension of Holmstrom and Milgrom (1987) with N agents. The principal has N production tasks one for each agent who cannot observe each other.

7 We assume σ satisfies a uniform Lipschitz condition: there exists a constant K such that for $Z, \bar{Z} \in C[0, 1]$, $|\sigma(t, Z) - \sigma(t, \bar{Z})| \le K \sup_{0\le s\le t} |Z(s) - \bar{Z}(s)|$. Even though this condition may be weakened (as was suggested by an anonymous referee) by noticing that our process is one dimensional and by employing Revuz and Yor (1999, Theorem 3.5, p. 390; Exercises 3.13–14, p. 397) (while it would still hold for the optimal contract), we use this Lipschitz condition (so, Revuz and Yor, 1999, Theorem 2.1, p. 375) in order to have a parallel presentation with Sung (1995).

8 This formulation is consistent with our hypothesis of the mean and variance being unobservable and nonverifiable by the principal. If $(S_i)_{i\in N}$ were to depend on the entire process $\{X_t\}_t$, implying that $\{X_t\}_t$ is observable and verifiable by the principal, then she could infer $\{\mu_t\}_t$ and/or $\{\sigma_t\}_t$. For more, see footnotes 7 and 8 of Sung (1995).

the beginning of the project. Any communication between the principal and the agents is not allowed as in Che and Yoo (2001).[9] Thus, at any instant a *utilitarian bargaining problem* among the agents emerges due to collusion opportunities. Its outcome can be implemented via state-contingent binding contracts drafted and agreed upon in date 0, specifying an allocation among the agents for each possible date and state. As for any given history agents' arrangement ensures optimality from that state onwards, our formulation involves renegotiation.

Collusion implies that the outcome of agents' bargaining is ex-ante efficient; so there is no history, state, and any other feasible contract that every agent (strictly) prefers to the one that was agreed upon. This brings about optimal risk sharing. Given $\mathcal{F}_t$-predictable salaries $\mathbf{S}_i : [0,1] \times \Omega \to \Re$, $i \in N$, agent i's induced salary at time t under $\mathbf{S} \equiv (\mathbf{S}_i)_{i\in N}$ is $\mathbf{S}_i(t) : \Omega \to \Re$, denoting the salary arrangement (on compensations to be made at the end of the project) to i under $\mathbf{S}$ at t. Below we define the agents' problem where the first requirement is a natural feasibility constraint, the second a balanced budget condition, and the third agent i's date-t participation constraint.[10]

Definition 1 (*The Agents' Problem*)**.** Given the principal's offer, salaries $S_i : \Omega \to \Re$ for $i \in N$ and $\mathcal{F}_t$-predictable control laws $\mu : [0,1] \times \Omega \to U$ and $\sigma : [0,1] \times \Omega \to \mathbb{S}$ and bargaining weights $\theta : [0,1] \times \Omega \to \text{int}(\Delta)$ (where $\text{int}(\Delta)$ denotes the interior of the N dimensional simplex), the side-contracting via control laws $\tilde{\mathbf{S}}_i : [0,1] \times \Omega \to \Re$ for $i \in N$, $\tilde{\mu} : [0,1] \times \Omega \to U$ and $\tilde{\sigma} : [0,1] \times \Omega \to \mathbb{S}$ solves the *agents' problem* at θ if for a.e. t and h^t

$$\sum_{i\in N} \theta_{it} E\left[-\exp\left\{ -r_i W_i^{\hat{\mathbf{S}}(t)}(X; \hat{\mu}, \hat{\sigma}) \right\} \Big| \mathcal{F}_t \right] \quad (1)$$

is maximized where $W_i^{\hat{\mathbf{S}}(t)}(X; \hat{\mu}, \hat{\sigma}) \equiv \left(\hat{\mathbf{S}}_i(t)(X) - \int_0^1 c_i(\hat{\mu}_s, \hat{\sigma}_s) ds \right)$, $X \in \Omega$, and

$$dX_\tau = \hat{\mu}_\tau d\tau + \hat{\sigma}_\tau dB_\tau, \quad \tau \ge t, \quad (2)$$

$$\sum_{i=1}^N \hat{\mathbf{S}}_i(t)(X) \le \sum_{i=1}^N S_i(X), \quad X \in \Omega, \quad (3)$$

$$E\left[-\exp\left\{ -r_i W_i^{\hat{\mathbf{S}}(t)}(X; \hat{\mu}, \hat{\sigma}) \right\} \Big| \mathcal{F}_t \right] \ge E\left[-\exp\left\{ -r_i W_i^S(X; \mu, \sigma) \right\} \big| \mathcal{F}_t \right], \quad i \in N. \quad (4)$$

The principal is aware of the collusion capabilities and bargaining among agents. Hence, she knows that while she is restricted to offer contracts that solve the agents' problem starting any date and state for some bargaining weights, she is not aware of agents' "real" bargaining weights.

Definition 2 (*The Principal's Problem*)**.** Principal chooses salary functions $\hat{S}_i : \Omega \to \Re$ for $i \in N$ and control laws $\hat{\mu} : [0,1] \times \Omega \to U$ and $\hat{\sigma} : [0,1] \times \Omega \to \mathbb{S}$ such that

$$\left((\hat{S}_i)_{i\in N}, \hat{\mu}, \hat{\sigma} \right) \in \underset{((S_i)_{i\in N}, \mu, \sigma)}{\text{argmax}} \times E\left[-\exp\left\{ -R\left(X_1 - \sum_{i=1}^N S_i(X) \right) \right\} \Big| \mathcal{F}_0 \right]$$

subject to

i. Feasibility: $dX_t = \mu_t dt + \sigma_t dB_t$, $t \in [0,1]$;
ii. Individual Rationality: $E\left[-\exp\left\{ -r_i W_i^S(X; \mu, \sigma) \right\} \big| \mathcal{F}_0 \right] \ge -\exp\{-r_i W_{i0}\}$, $i \in N$;
iii. The Agents' Problem: $(S_i)_{i\in N}$ and (μ, σ) must be such that there exists a profile of control laws $\mathbf{S}_i : [0,1] \times \Omega \to \Re$ satisfying $\mathbf{S}_i(1)(X) = S_i(X)$, $i \in N$ and $X \in \Omega$, so that $((\mathbf{S})_i, \mu, \sigma)$ solves the agents' problem at some bargaining weights $\theta : [0,1] \times \Omega \to \text{int}(\Delta)$ given $((S_i)_i, \mu, \sigma)$.

In Definition 2, feasibility and individual rationality are standard. Collusion, on the other hand, is handled by requiring that the principal's offer solves the agents' problem.

3. Optimality of linearity

Our main theorem proves that the linearity results of Holmstrom and Milgrom (1987), Schättler and Sung (1993), and Sung (1995) are robust with respect to collusion and renegotiation.

Theorem 1. *There exists a stationary and linear optimal collusion proof and renegotiation proof contract.*

The rest of the paper concerns the proof of this result which involves 3 steps. First, we analyze the efficiency implications of the agents' problem and obtain some desirable properties. In fact, we show that the interaction among agents is similar to that in Bone (1998) and its aggregation result holds in our setting. This enables us to associate the agents' problem with one that involves a "representative agent" (the team of all agents) having a CARA utility.[11,12] In the second step we consider the associated version of the principal's problem with a team and establish optimality of linearity as in Sung (1995). The final step shows that this result is preserved in the principal's problem containing the agents' when the team's payments are distributed efficiently.

Definition 3. Given t and h^t and $(\mu_\tau, \sigma_\tau)_{\tau\in[0,1]}$, $(\mathbf{S}_i(t))_{i\in N}$ is *static efficient at* h^t if there exists $\theta_t \in \text{int}(\Delta)$ such that

$$(\mathbf{S}_i(t))_{i\in N} \in \arg\max_{\hat{\mathbf{S}}(t)} \sum_{i\in N} \theta_{it} E\left[-\exp\left\{ -r_i W_i^{\hat{\mathbf{S}}(t)}(X; \mu, \sigma) \right\} \Big| \mathcal{F}_t \right]$$

subject to (1) $dX_\tau = \mu_\tau d\tau + \sigma_\tau dB_\tau$ for $\tau \ge t$, and (2) $\sum_{i=1}^N \hat{\mathbf{S}}_i(t)(X) \le \sum_{i=1}^N \mathbf{S}_i(t)(X)$ for $X \in \Omega$.

[9] Otherwise by offering additional payoffs the principal can make the agents report others' choices and implement the first-best, at least in a one-shot setting. While the value of this communication is not trivial due to agents' abilities to punish "snitches" in a repeated setting, not allowing any communication between the principal and the agents helps us to abstract from these complications. For more see footnote 13 of Barlo and Özdoğan (2013). Moreover, footnote 2 of the current paper provides examples when this abstraction is plausible.

[10] The date-t participation constraint considers the grand coalition/team, and not sub-coalitions. This can be justified when one assumes that each player has a right to veto the outcome of the agents' bargaining.

[11] In that study a group of agents with CARA utilities jointly choose between uncertain prospects. A static environment is modeled, while the following two key aspects are common with our setting: (1) the choice of any prospect must be unanimously agreed, and (2) the uncertain outcomes from the chosen prospect are distributed among agents according to some unanimously made prior agreements.

[12] An earlier study, Brennan and Kraus (1978), shows that an aggregation leading to a representative agent representation is possible when agents have either CARA utilities or HARA (hyperbolic absolute risk aversion) preferences with equal exponents. And, it is shown in Section 4 in Bone (1998) that this conclusion does not hold with nonidentical exponents. Moreover, the representative agent's utility function is not necessarily negative exponential with HARA utilities having identical exponents. However, as the stationary decision making environment is a key feature in the search for optimality of linearity, the CARA utilities' property of not involving any income effects and the use of stochastic processes with the martingale property are essential: the history in our setting determines the accumulated returns which do not influence agents' decisions due to lack of income effects; and, incremental future returns is expected not to be different from today's due to the martingale property.

Lemma 1. *Given t and h^t and $(\mu_\tau, \sigma_\tau)_{\tau\in[0,1]}$, $(\mathbf{S}_i(t))_{i\in N}$ is static efficient at h^t if and only if there is $(\theta_{it})_{i\in N} \in \text{int}(\Delta)$ such that for a.e. $X \in \Omega$ and for any $i, j \in N$*

$$\theta_{it} r_i \exp\left\{-r_i W_i^{\mathbf{S}(t)}(X;\mu,\sigma)\right\} = \theta_{jt} r_j \exp\left\{-r_j W_j^{\mathbf{S}(t)}(X;\mu,\sigma)\right\}. \quad (5)$$

Proof. Let t and h^t and $(\mu_\tau, \sigma_\tau)_{\tau\in[0,1]}$ be given. We denote the resulting probability density function on X by $f(\cdot;\mu,\sigma \mid \mathcal{F}_t)$.[13] Due to the strict concavity of the utility functions, the necessary conditions of the first-order analysis are also sufficient. Hence, $(\mathbf{S}_i(t))_{i\in N}$ is static efficient at h^t if and only if for every $i \in N$

$$\theta_{it} r_i \exp\left\{-r_i W_i^{\mathbf{S}(t)}(X;\mu,\sigma)\right\} f(X;\mu,\sigma \mid \mathcal{F}_t) = \lambda_X, \quad \text{for a.e. } X \in \Omega \quad (6)$$

where λ_X denotes the Lagrangian multiplier of the feasibility for the redistribution in state X and it has to be strictly positive as the constraint binds due to the objective function being strictly increasing. Note that $f(X;\mu,\sigma \mid \mathcal{F}_t) > 0, X \in \Omega$, and (6) is analogous to condition 4 of Bone (1998). Since the right-hand side of (6) does not depend on the identity of the agent, the result follows. ■

By following the same arithmetic manipulations of Bone (1998) (conditions 9–13), we obtain:

Lemma 2. *Given t and h^t and $(\mu_\tau, \sigma_\tau)_{\tau\in[0,1]}$, $(\mathbf{S}_i(t))_{i\in N}$ is static efficient at h^t if and only if there is $(\theta_{it})_{i\in N} \in \text{int}(\Delta)$ such that for all $i \in N$ and a.e. $X \in \Omega$,*

$$W_i^{\mathbf{S}(t)}(X;\mu,\sigma) = k_{it} + \frac{r_c}{r_i}\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma) \quad (7)$$

where $r_c = (\sum_j 1/r_j)^{-1}$ and $k_{it} = (r_c/r_i)(\sum_j (\ln(\theta_{it} r_i) - \ln(\theta_{jt} r_j))/r_j)$ and $\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma) = \sum_i W_i^{\mathbf{S}(t)}(X;\mu,\sigma)$.

Proof. The rearrangement of Eq. (5) in logarithmic form is as follows: for a.e. $X \in \Omega$ and every $i, j \in N$, there exist $\{\theta_{it}, \theta_{jt}\}$ at time t such that,

$$W_j^{\mathbf{S}(t)}(X;\mu,\sigma) = \frac{r_i}{r_j} W_i^{\mathbf{S}(t)}(X;\mu,\sigma) + \frac{\ln(\theta_{jt} r_j) - \ln(\theta_{it} r_i)}{r_j}.$$

Summing across j while keeping i fixed results in

$$\begin{aligned}\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma) &= \sum_j W_j^{\mathbf{S}(t)}(X;\mu,\sigma) \\ &= \sum_j \left(\frac{r_i}{r_j} W_i^{\mathbf{S}(t)}(X;\mu,\sigma) + \frac{\ln(\theta_{jt} r_j) - \ln(\theta_{it} r_i)}{r_j}\right) \\ &= r_i W_i^{\mathbf{S}(t)}(X;\mu,\sigma) \sum_j \frac{1}{r_j} + \sum_j \frac{\ln(\theta_{jt} r_j) - \ln(\theta_{it} r_i)}{r_j} \\ &= \frac{r_i}{r_c} W_i^{\mathbf{S}(t)}(X;\mu,\sigma) - \frac{r_i}{r_c} k_{it}\end{aligned}$$

where r_c and k_{it} are as defined in the statement of the lemma. Hence, the result follows. ■

So given the history and control laws, static efficiency at that history implies that agent i's payment in instant t from the total payments (the team's state-contingent compensation) involves a (state-independent) constant payment, and a fraction which depends on agents' CARA coefficients and not the bargaining weights. Moreover, summing across agents these fractions add up to unity while the fixed payments sum to zero. This leads to the following:

Lemma 3. *Suppose that for given t and h^t and $(\mu_\tau, \sigma_\tau)_{\tau\in[0,1]}$, $(\mathbf{S}_i(t))_{i\in N}$ is static efficient at h^t and let $\theta_t \in \text{int}(\Delta)$, identifying $(k_{it})_i \in \Re^N$ according to Lemma 2, be associated with $(\mathbf{S}_i(t))_i$. Let $\mathbf{Q}(t) : \Omega \to \Re$ be given and $\bar{W}^{\mathbf{Q}(t)}(X;\mu,\sigma) \equiv \left(\mathbf{Q}(t)(X) - \sum_i\left(\int_0^1 c_i(\mu_s, \sigma_s)ds\right)\right)$, $X \in \Omega$. Then $(\tilde{\mathbf{S}}_i(t))_{i\in N}$, a feasible redistribution of $\mathbf{Q}(t)$ according to θ_t (thus, $(k_{it})_i$) defined by*

$$W_i^{\tilde{\mathbf{S}}(t)}(X;\mu,\sigma) = k_{it} + \frac{r_c}{r_i}\bar{W}^{\mathbf{Q}(t)}(X;\mu,\sigma),$$

for a.e. $X \in \Omega$, is also static efficient at h^t.

Proof. Let t and h^t and $(\mu_\tau, \sigma_\tau)_{\tau\in[0,1]}$ be given, and $(\mathbf{S}_i(t))_{i\in N}$ along with θ_t and $(k_{it})_i$ be as in the statement of the lemma. Hence, due to Lemmas 1 and 2 the profile $(\mathbf{S}_i(t))_{i\in N}$ is static efficient at h^t is equivalent to for a.e. $X \in \Omega$

$$\begin{aligned}&\theta_{it} r_i \exp\left\{-r_i\left(k_{it} + \frac{r_c}{r_i}\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma)\right)\right\} \\ &\quad = \theta_{jt} r_j \exp\left\{-r_j\left(k_{jt} + \frac{r_c}{r_j}\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma)\right)\right\},\end{aligned}$$

which simplifies to, $\theta_{it} r_i \exp\{-r_i k_{it}\} = \theta_{jt} r_j \exp\{-r_j k_{jt}\}$. To see that $(\tilde{\mathbf{S}}_i(t))_{i\in N}$ is static efficient at h^t we prove that this profile satisfies (5). This follows from the last equation and for a.e. $X \in \Omega$ we have $\bar{W}^{\tilde{\mathbf{S}}(t)}(X;\mu,\sigma) = \bar{W}^{\mathbf{Q}(t)}(X;\mu,\sigma)$ and

$$\frac{\theta_{it} r_i}{\theta_{jt} r_j} = \frac{\exp\{-r_j k_{jt}\}}{\exp\{-r_i k_{it}\}} = \frac{\exp\left\{-r_j\left(k_{jt} + \frac{r_c}{r_j}\bar{W}^{\tilde{\mathbf{S}}(t)}(X;\mu,\sigma)\right)\right\}}{\exp\left\{-r_i\left(k_{it} + \frac{r_c}{r_i}\bar{W}^{\tilde{\mathbf{S}}(t)}(X;\mu,\sigma)\right)\right\}}. \quad ■$$

We employ this lemma to establish that the principal does need not to know what the "real" bargaining weights θ_t are. As the bargaining weights do not affect agents' shares from the total compensation when dealing with static efficiency at a given history, it can be shown that in such situations the interests of all the agents are perfectly aligned.

Lemma 4. *Suppose that for given t and h^t and $(\mu_\tau, \sigma_\tau)_{\tau\in[0,1]}$, $(\mathbf{S}_i(t))_{i\in N}$ associated with θ_t and $(k_{it})_i$ and $\left(\mathbf{S}'_i(t)\right)_{i\in N}$ are both static efficient at h^t with the additional requirement that $\left(\mathbf{S}'_i(t)\right)_{i\in N}$ is defined by*

$$W_i^{\mathbf{S}'(t)}(X;\mu',\sigma') = k_{it} + \frac{r_c}{r_i}\bar{W}^{\mathbf{S}'(t)}(X;\mu',\sigma').$$

Then,

$$\begin{aligned}&E\left[-\exp\left\{-r_j W_j^{\mathbf{S}(t)}(X;\mu,\sigma)\right\}\Big|\,\mathcal{F}_t\right] \\ &\quad > E\left[-\exp\left\{-r_j W_j^{\mathbf{S}'(t)}(X;\mu',\sigma')\right\}\Big|\,\mathcal{F}_t\right], \quad \text{for some } j \in N \quad (8)\end{aligned}$$

if and only if

$$\begin{aligned}&E\left[-\exp\left\{-r_c \bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma)\right\}\Big|\,\mathcal{F}_t\right] \\ &\quad > E\left[-\exp\left\{-r_c \bar{W}^{\mathbf{S}'(t)}(X;\mu',\sigma')\right\}\Big|\,\mathcal{F}_t\right]. \quad (9)\end{aligned}$$

Proof. Let t and h^t and $(\mu_\tau, \sigma_\tau)_{\tau\in[0,1]}$ be given and $(\mathbf{S}_i(t))_{i\in N}$ associated with θ_t and $(k_{it})_i$ and $\left(\mathbf{S}'_i(t)\right)_{i\in N}$ be as in the statement of the lemma. Notice that in light of Lemma 2 (Eq. (7)), inequality (8) holds for any one of $j \in N$ if and only if

[13] It is useful to remind the reader that for any *standard Brownian motion* $\mathbf{X} = \{X_t : t \in [0,\infty)\}$, X_t has a probability density function f_t given by $f_t(x) = 1/(\sqrt{2\pi t})\exp\{-x^2/(2t)\}$.

$$E\left[-\exp\left\{-r_j\left(k_{jt}+\frac{r_c}{r_j}\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma)\right)\right\}\middle|\mathcal{F}_t\right]$$
$$>E\left[-\exp\left\{-r_j\left(k_{jt}+\frac{r_c}{r_j}\bar{W}^{\mathbf{S}'(t)}(X;\mu',\sigma')\right)\right\}\middle|\mathcal{F}_t\right]$$

which equivalent to

$$\exp\{-r_jk_{jt}\}E\left[-\exp\left\{r_c\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma)\right\}\middle|\mathcal{F}_t\right]$$
$$>\exp\{-r_jk_{jt}\}E\left[-\exp\left\{r_c\bar{W}^{\mathbf{S}'(t)}(X;\mu',\sigma')\right\}\middle|\mathcal{F}_t\right],$$

delivering the desired conclusion as the last inequality is equivalent to (9). ∎

Now, we proceed with associating these conclusions with dynamic notions of efficiency:

Definition 4. Given $(\mu_\tau,\sigma_\tau)_{\tau\in[0,1]}$, $(\mathbf{S}_i)_{i\in N}$ with $\mathbf{S}_i:[0,1]\times\Omega\to\Re$ for $i\in N$ is efficient if for a.e. t and h^t it must be that $(\mathbf{S}_i(t))_{i\in N}$ is static efficient at h^t. We say that $((\mathbf{S}_i)_i,\mu,\sigma)$ is efficient whenever $(\mathbf{S}_i)_i$ is efficient for given μ and σ.

Then, under the light of our findings about efficiency and the fact that agents' interest are perfectly aligned, we wish to define *the team's problem*:

Definition 5 (*The Team's Problem*)**.** Given the principal's offer, salaries $S_i:\Omega\to\Re$ for $i\in N$ and $\mathcal{F}_t$-predictable control laws $\mu:[0,1]\times\Omega\to U$ and $\sigma:[0,1]\times\Omega\to\mathbb{S}$, $\tilde{\mathbf{S}}_c:[0,1]\times\Omega\to\Re$ and $\tilde{\mu}:[0,1]\times\Omega\to U$ and $\tilde{\sigma}:[0,1]\times\Omega\to\mathbb{S}$ solve the *team's problem* if for a.e. t and h^t the following is maximized

$$E\left[-\exp\left\{-r_cW_c^{\hat{\mathbf{S}}_c(t)}(X;\hat{\mu},\hat{\sigma})\right\}\middle|\mathcal{F}_t\right],\tag{10}$$

where $W_c^{\hat{\mathbf{S}}_c(t)}(X;\hat{\mu},\hat{\sigma})\equiv\left(\hat{\mathbf{S}}_c(t)(X)-\sum_i\left(\int_0^1c_i(\hat{\mu}_s,\hat{\sigma}_s)\mathrm{d}s\right)\right)$, $X\in\Omega$, subject to

$$\mathrm{d}X_\tau=\hat{\mu}_\tau\mathrm{d}\tau+\hat{\sigma}_\tau\mathrm{d}B_\tau,\quad\tau\geq t,\tag{11}$$

$$\hat{\mathbf{S}}_c(t)(X)\leq\sum_iS_i(X),\quad X\in\Omega.\tag{12}$$

$$E\left[-\exp\left\{-r_cW_c^{\hat{\mathbf{S}}_c(t)}(X;\hat{\mu},\hat{\sigma})\right\}\middle|\mathcal{F}_t\right]$$
$$\geq E\left[-\exp\left\{-r_iW_i^S(X;\hat{\mu},\hat{\sigma})\right\}\middle|\mathcal{F}_t\right],\quad\forall i\in N.\tag{13}$$

The date-t participation constraint, (13), can be interpreted as follows: the expected utility of the representative agent (the team) cannot be strictly lower than the expected utility of any one of the agents. Otherwise whether or not such an agent would be willing to participate into the team arrangement is at jeopardy.

For any control laws $(\mathbf{S}^T,\mu^T,\sigma^T)$ that solve the team's problem, we prove that we can construct a redistribution so that efficiency in every date and state is obtained and agents' problem is solved.

Lemma 5. *Let the principal's offer, $S_i:\Omega\to\Re$, $i\in N$, and $\mathcal{F}_t$-predictable $\mu:[0,1]\times\Omega\to U$ and $\sigma:[0,1]\times\Omega\to\mathbb{S}$ be given, and suppose that the $\mathcal{F}_t$-predictable profile $(\mathbf{S}_c^T,\mu^T,\sigma^T)$, with $\mathbf{S}_c^T:[0,1]\times\Omega\to\Re$ and $\mu^T:[0,1]\times\Omega\to U$ and $\sigma^T:[0,1]\times\Omega\to\mathbb{S}$, solves the team's problem. Then $((\mathbf{S}_i^T)_i,\mu^T,\sigma^T)$ obtained by distributing the team's payments with $\theta^*:[0,1]\times\Omega\to\mathrm{int}(\Delta)$ where $\theta_{it}^*=\frac{r_c}{r_i}$ for all t and h^t is efficient and solves the agents' problem at θ^* for given $((S_i)_i,\mu,\sigma)$.*

Proof. Let the principal's offer $((S_i)_i,\mu,\sigma)$ be as in the statement of the lemma and suppose $(\mathbf{S}_c^T,\mu^T,\sigma^T)$ solves the team's problem. Define $\mathbf{S}^T=(\mathbf{S}_i^T)_i$ using θ^* as follows: for a.e. t, h^t and X

$$W_i^{\mathbf{S}^T(t)}(X;\mu^T,\sigma^T)=\frac{r_c}{r_i}\bar{W}^{\mathbf{S}^T(t)}(X;\mu^T,\sigma^T),\tag{14}$$

while $\bar{W}^{\mathbf{S}^T(t)}(X;\mu^T,\sigma^T)=W_c^{\mathbf{S}_c^T(t)}(X;\mu^T,\sigma^T)$ for a.e. t and h^t and X.

Next, we prove that $((\mathbf{S}_i^T)_i,\mu^T,\sigma^T)$ is efficient: let t and h^t be given and $\theta_i^*=\frac{r_c}{r_i}$. Observe that $k_{it}^T=(r_c/r_i)(\sum_j(\ln(\theta_i^*r_i)-\ln(\theta_j^*r_j))/r_j)=0$ for all i,t,h^t. So (14), the defining condition of $(\mathbf{S}_i^T(t))_i$, satisfies (7); hence, $(\mathbf{S}_i^T(t))_i$ is static efficient at h^t by Lemma 2.

$(\mathbf{S}_c^T,\mu^T,\sigma^T)$ solving the team's problem means that for a.e. t and h^t it maximizes (10) subject to (11) and (12) and (13). We wish to show that $((\mathbf{S}_i^T)_i,\mu^T,\sigma^T)$ satisfies the constraints of the agents' problem. As $\sum_i\frac{r_c}{r_i}=1$, (11) and (12) imply (2) and (3). We display that (4) also holds: since $(\mathbf{S}_i^T)_i$ is defined for a.e. t and h^t and X by (14) it must be that $r_iW_i^{\mathbf{S}^T(t)}(X;\mu^T,\sigma^T)=r_c\bar{W}^{\mathbf{S}^T(t)}(X;\mu^T,\sigma^T)$ and $\bar{W}^{\mathbf{S}^T(t)}(X;\mu^T,\sigma^T)=W_c^{\mathbf{S}_c^T(t)}(X;\mu^T,\sigma^T)$ for a.e. t and h^t and X; this implies (13) if and only if (4).

In the next step we prove that for any $((\mathbf{S}_i^A)_i,\mu^A,\sigma^A)$ that solve the agents' problem for θ^*, the associated profile $(\sum_i\mathbf{S}_i^A,\mu^A,\sigma^A)$ satisfies (11) and (12) and (13) of the team's problem. Notice that $((\mathbf{S}_i^A)_i,\mu^A,\sigma^A)$ is efficient: since the definition of static efficiency concerns the maximization of (1) subject to (2) and (3) for a given t and h^t and $(\mu_\tau^A,\sigma_\tau^A)_{\tau\in[0,1]}$, we conclude that for a.e. t and h^t, $((\mathbf{S}_i^A(t))_i,\mu^A,\sigma^A)$ is static efficient at h^t. So Lemma 2 applies and using θ^* we obtain:

$$W_i^{\mathbf{S}^A(t)}(X;\mu^A,\sigma^A)=k_{it}^A+\frac{r_c}{r_i}\bar{W}^{\mathbf{S}^A(t)}(X;\mu^A,\sigma^A),$$
$$\text{where }k_{it}^A=0\text{ for all }t\text{ and }h^t\text{ and }i.\tag{15}$$

(2) and (3) concerning $((\mathbf{S}_i^A)_i,\mu^A,\sigma^A)$ imply (11) and (12) involving $(\sum_i\mathbf{S}_i^A,\mu^A,\sigma^A)$. And (13) if and only if (4): since $(\mathbf{S}_i^A)_i$ is defined for a.e. t and h^t and X by (15), $r_iW_i^{\mathbf{S}^A(t)}(X;\mu^A,\sigma^A)=r_c\bar{W}^{\mathbf{S}^A(t)}(X;\mu^A,\sigma^A)$ and $\bar{W}^{\mathbf{S}^A(t)}(X;\mu^A,\sigma^A)=W_c^{\bar{\mathbf{S}}^A(t)}(X;\mu^A,\sigma^A)$ where $\bar{\mathbf{S}}^A(t)(X)=\sum_i\mathbf{S}_i^A(t)(X)$, $i\in N$.

The preceding two paragraphs establish that (1) the solution to the team's problem satisfies the constraints of the agents' problem when the distribution is done according to θ^*, and (2) the solution of the agents' problem at θ^* satisfies the constraints of the team's problem.

Finally, we establish that if $(\mathbf{S}_c^T,\mu^T,\sigma^T)$ solves the team's problem, then $((\mathbf{S}_i^T)_i,\mu^T,\sigma^T)$ solves the agents' problem at θ^*. From the above we know that $((\mathbf{S}_i^T)_i,\mu^T,\sigma^T)$ is efficient. So using (14) and θ^*, the objective function of the agents' problem (condition (1)) becomes

$$\sum_i\theta_i^*E\left[-\exp\{-r_iW_i^{\mathbf{S}^T(t)}(X;\mu,\sigma)\}\middle|\mathcal{F}_t\right]$$
$$=\sum_i\frac{r_c}{r_i}E\left[-\exp\{-r_c\bar{W}^{\mathbf{S}^T(t)}(X;\mu,\sigma)\}\middle|\mathcal{F}_t\right]$$
$$=r_c\left(\sum_i\frac{1}{r_i}\right)E\left[-\exp\{-r_cW_c^{\mathbf{S}_c^T(t)}(X;\mu,\sigma)\}\middle|\mathcal{F}_t\right]$$
$$=E\left[-\exp\left\{-r_cW_c^{\mathbf{S}_c^T(t)}(X;\mu,\sigma)\right\}\middle|\mathcal{F}_t\right].$$

Therefore, the objective functions of the two problems coincide, delivering the desired conclusion. ∎

Now, the principal may contract directly with the representative agent having a CARA coefficient $r_c=\left(\sum_i\frac{1}{r_i}\right)^{-1}$ and a reservation certainty equivalent $W_{c0}=\sum_iW_{i0}$ and costs $c_c:U\times\mathbb{S}\to\Re$ is defined by $c_c(\mu_t,\sigma_t)=\sum_ic_i(\mu_t,\sigma_t)$:

Definition 6. Principal chooses a salary for the team $\hat{S}_c : \Omega \to \Re$ and control laws $\hat{\mu} : [0,1] \times \Omega \to U$ and $\hat{\sigma} : [0,1] \times \Omega \to \mathbb{S}$, such that

$$\left(\hat{S}_c, \hat{\mu}, \hat{\sigma}\right) \in \underset{(S_c,\mu,\sigma)}{\operatorname{argmax}} E\left[-\exp\left\{-R\left(X_1 - S_c(X)\right)\right\} \mid \mathcal{F}_0\right]$$

subject to

i. $dX_t = \mu_t dt + \sigma_t dB_t, t \in [0,1]$;
ii. $E\left[-\exp\left\{-r_c W_c^{S_c}(X;\mu,\sigma)\right\} \middle| \mathcal{F}_0\right] \geq -\exp\{-r_c W_{c0}\}$;
iii. S_c and (μ,σ) must be such that there exists $\mathbf{S}_c : [0,1] \times \Omega \to \Re$ satisfying $\mathbf{S}_c(1)(X) = S_c(X), X \in \Omega$, so that for a.e. t and h^t, $(\mathbf{S}_c, \mu, \sigma)$ maximizes $E\left[-\exp\left\{-r_c W_c^{\tilde{\mathbf{S}}_c(t)}(X;\tilde{\mu},\tilde{\sigma})\right\} \middle| \mathcal{F}_t\right]$ subject to $dX_\tau = \tilde{\mu}_\tau d\tau + \tilde{\sigma}_\tau dB_\tau, \tau \geq t$, and $\tilde{\mathbf{S}}_c(t)(X) \leq S_c(X)$, $X \in \Omega$.

The principal's problem involving the representative agent given in Definition 6 belongs to the class studied in Sung (1995) and his Proposition 2 applies which we restate using our notation.[14]

Lemma 6 (*Proposition 2 of Sung, 1995*)**.** *Let (m^*, s^*) be a control pair that solves the following constrained static maximization problem. Choose $(\hat{m}, \hat{s}) \in U \times \mathbb{S}$ to maximize*

$$\Phi^p(\hat{m},\hat{s}) = \hat{m} + R\hat{s}^2 c_{c\mu}(\hat{m},\hat{s}) - c_c(\hat{m},\hat{s}) - \frac{1}{2}(R + r_c)\left(c_{c\mu}(\hat{m},\hat{s})\right)^2 \hat{s}^2 - \frac{R}{2}\hat{s}^2$$

subject to $(\hat{m},\hat{s}) \in \arg\max_{(m,s)\in U\times\mathbb{S}} \Phi^a(m, s \mid \hat{m},\hat{s}) := c_{c\mu}(\hat{m},\hat{s})m - c_c(m,s) - \frac{r_c}{2}(c_{c\mu}(\hat{m},\hat{s}))^2 s^2$.

Then (m^*, s^*) is the optimal control pair for all $t \in [0,1]$, and the principal's optimal remaining expected utility V over time is given by $V(t, X_t) = -\exp\left\{-R\left(X_t - W_{c0} + (1-t)\Phi^p(m^*,s^*)\right)\right\}$.

Furthermore, the optimal salary scheme S_c^* is linear in the final realized outcome X_1, and is given by

$$S_c^*(X_1) = W_{c0} + c_c(m^*,s^*) + c_{c\mu}(m^*,s^*)\left((X_1 - X_0) - m^*\right) + \frac{r_c}{2}\left(c_{c\mu}(m^*,s^*)\right)^2 s^{*2}. \quad (16)$$

Proof. See the Appendix of Sung (1995). ■

We have to emphasize that "Φ^p is representative of the principal's expected utility" while "Φ^a can be viewed as a representative of the (representative) agent's expected utility" (Sung, 1995). Therefore, Lemma 6 tells that the principal's problem given in Definition 6 has a (stationary) solution (S_c^*, μ^*, σ^*) where $\mu^* : [0,1] \times \Omega \to U$ and $\sigma^* : [0,1] \times \Omega \to \mathbb{S}$ are defined by $\mu_t^*(X) = m^*$ and $\sigma_t^*(X) = s^*$ for $t \in [0,1]$ and $X \in \Omega$, and S_c^* is linear in X_1 as it is given by Eq. (16).

The principal distributing (S_c^*, μ^*, σ^*), efficiently using θ^* attains $S^* = (S_i^*)_i$ defined by

$$W_i^{S^*}(X;\mu^*,\sigma^*) = (r_c/r_i) W_c^{S_c^*}(X;\mu^*,\sigma^*). \quad (17)$$

Let $\mathbf{S}_k^* : [0,1] \times \Omega \to \Re$ for $k = c, 1, \ldots, N$ be given by $\mathbf{S}_k^*(t)(X) = S_k^*(X), t \in [0,1]$ and $X \in \Omega$.

Lemma 7. *$S_i^* : \Omega \to \Re$ is linear in X_1 for all $i \in N$. And $(\mathbf{S}_c^*, \mu^*, \sigma^*)$ solves the team's problem (Definition 5) given $((S_i^*)_i, \mu^*, \sigma^*)$; $((\mathbf{S}_i^*)_i, \mu^*, \sigma^*)$ solves the agents' problem (Definition 1) given $((S_i^*)_i, \mu^*, \sigma^*)$ at θ^*. Finally, $((S_i^*)_i, \mu^*, \sigma^*)$ solves the principal's problem (Definition 2).*

Proof. The linearity of S_i^* follows from the fact that (17) is equivalent to $S_i^*(X)$ being equal to

$$S_i^*(X) = c_i(m^*,s^*) + \frac{r_c}{r_i}\left(\sum_{j\in N} W_{j0} + A_1(X_1 - X_0) + A_2\right),$$

where $A_1 = \sum_{j\in N} c_{j\mu}(m^*,s^*)$ and $A_2 = \frac{r_c}{2}(\sum_{j\in N} c_{j\mu}(m^*,s^*))^2 s^{*2} - (\sum_{j\in N} c_{j\mu}(m^*,s^*))m^*$. As $c_{i\mu}$ is strictly positive, A_1 is strictly positive.

To establish that $(\mathbf{S}_c^*, \mu^*, \sigma^*)$ solves the team's problem given $((S_i^*)_i, \mu^*, \sigma^*)$, it suffices to show that (13) is satisfied. This holds because by definition $E[-\exp\{-r_c W_c^{\mathbf{S}_c^*(t)}(X;\mu^*,\sigma^*)\}|\mathcal{F}_t]$ equals $E[-\exp\{-r_c W_c^{S_c^*}(X;\mu^*,\sigma^*)\}|\mathcal{F}_t] = E[-\exp\{-r_i W_i^{S_i^*}(X;\mu^*,\sigma^*)\}|\mathcal{F}_t]$, $i \in N$, due to (17).

Now, Lemma 5 applies, so $((\mathbf{S}_i^*)_i, \mu^*, \sigma^*)$ solves the agents' problem given $((S_i^*)_i, \mu^*, \sigma^*)$ at θ^*.

To show that $((S_i^*)_i, \mu^*, \sigma^*)$ solves the principal's problem given in Definition 2 it suffices to prove that this profile satisfies agents' individual rationality constraints. This follows from the fact that $E[-\exp\{-r_i(S_i^* - \int_0^1 c_i(\mu^*,\sigma^*)dt)\}|\mathcal{F}_0] = E[-\exp\{-r_i(W_{i0} + \frac{r_c}{r_i}(A_1(X_1 - X_0) + A_2))\}|\mathcal{F}_0]$, and this equals $-\exp\{-r_i W_{i0}\}$ $E[-\exp\{-r_c(S_c^* - W_{c0} - c_c(m^*,s^*))\}|\mathcal{F}_0]$, and the individual rationality constraint of the representative agent (condition ii in Definition 6) being satisfied. ■

This finishes the proof of Theorem 1.

4. Concluding remarks

Now, we consider the situation when agents' "real" bargaining weights are employed. Let $\theta^R : [0,1] \times \Omega \to \text{int}(\Delta)$ be the agents' real bargaining weights that the principal is not aware of. Below we prove that $((\mathbf{S}_i^*)_i, \mu^*, \sigma^*)$ also solves the agents' problem given $((S_i^*)_i, \mu^*, \sigma^*)$ at θ^R.

Suppose not, and consider $((\mathbf{S}_i^R)_i, \mu^*, \sigma^*)$ where $\mathbf{S}_i^R$ is defined by

$$W_i^{\mathbf{S}^R(t)}(X;\mu^*,\sigma^*) = k_{it}^R + \frac{r_c}{r_i} W_c^{\mathbf{S}_c^*(t)}(X;\mu^*,\sigma^*), \quad (18)$$

while $(k_{it}^R)_i$ is associated with $(\theta_{it}^R)_i$. Due to Lemma 3 we know that $((\mathbf{S}_i^R)_i, \mu^*, \sigma^*)$ is efficient. If $((\mathbf{S}_i^R)_i, \mu^*, \sigma^*)$ were not to solve the agents' problem given $((S_i^*)_i, \mu^*, \sigma^*)$ at θ^R, then the solution $((\mathbf{S}_i^A)_i, \mu^A, \sigma^A)$ must be efficient, thus satisfy (7) with $(k_{it}^R)_i$ (i.e. is given by $W_i^{\mathbf{S}^A(t)}(X;\mu^A,\sigma^A) = k_{it}^R + \frac{r_c}{r_i} W_c^{\mathbf{S}_c^*(t)}(X;\mu^A,\sigma^A), i \in N$), and that there exist t and h^t with

$$\sum_i \theta_{it}^R E\left[-\exp\left\{-r_i W_i^{\mathbf{S}^A(t)}(X,\mu^A,\sigma^A)\right\} \middle| \mathcal{F}_t\right] > \sum_i \theta_{it}^R E\left[-\exp\left\{-r_i W_i^{\mathbf{S}^R(t)}(X,\mu^*,\sigma^*)\right\} \middle| \mathcal{F}_t\right],$$

which implies that there is some $j \in N$ such that $E[-\exp\{-r_j W_j^{\mathbf{S}^A(t)}(X,\mu^A,\sigma^A)\}|\mathcal{F}_t]$ strictly exceeds $E[-\exp\{-r_j W_j^{\mathbf{S}^R(t)}(X,\mu^*,\sigma^*)\}|\mathcal{F}_t]$. Since both $((\mathbf{S}_i^A)_i, \mu^A, \sigma^A)$ and $((\mathbf{S}_i^R)_i, \mu^*, \sigma^*)$ are efficient and defined via the same $(k_{it}^R)_i$, Lemma 4 applies and the last inequality is equivalent to $E[-\exp\{-r_c W_c^{\mathbf{S}_c^*(t)}(X,\mu^A,\sigma^A)\}|\mathcal{F}_t]$ being strict greater than $E[-\exp\{-r_c W_c^{\mathbf{S}_c^*(t)}(X,\mu^*,\sigma^*)\}|\mathcal{F}_t]$ and this delivers a contradiction to μ^* and σ^* being optimal controls of Lemma 6.

Having established that $((\mathbf{S}_i^R)_i, \mu^*, \sigma^*)$ solves the agents' problem for given $((S_i^*)_i, \mu^*, \sigma^*)$ at θ^R, we obtain from (18)

[14] Sung (1995) uses the first-order approach, introduced by Schättler and Sung (1993), by allowing agents to control the variance as well as the mean of the process. The first-order necessary conditions lead to a semi-martingale representation of agent's salary function which, in turn, is used to obtain a relaxed version of the principal's problem.

that $r_i W_i^{\mathbf{S}^R(t)}(X;\mu^*,\sigma^*) = r_i k_{it}^R + r_c W_c^{\mathbf{S}_c^*(t)}(X;\mu^*,\sigma^*)$, and use the observation that made in (14) to have $r_c W_c^{\mathbf{S}_c^*(t)}(X;\mu^*,\sigma^*) = r_i W_i^{\mathbf{S}^*(t)}(X;\mu^*,\sigma^*)$ delivering

$$k_{it}^R = W_i^{\mathbf{S}^R(t)}(X;\mu^*,\sigma^*) - W_i^{\mathbf{S}^*(t)}(X;\mu^*,\sigma^*).$$

Due to $W_i^{\mathbf{S}^*(t)}(X;\mu^*,\sigma^*) = W_i^{S^*}(X;\mu^*,\sigma^*)$, i's date-t participation constraint (4), becomes

$$\begin{aligned}
0 &\le E\left[-\exp\left\{-r_i W_i^{\mathbf{S}^R(t)}(X;\mu^*,\sigma^*)\right\}\Big|\,\mathcal{F}_t\right] \\
&\quad + E\left[\exp\left\{-r_i W_i^{S^*}(X;\mu^*,\sigma^*)\right\}\Big|\,\mathcal{F}_t\right] \\
&= \frac{E\left[-\exp\left\{-r_i W_i^{\mathbf{S}^R(t)}(X;\mu^*,\sigma^*)\right\}\Big|\,\mathcal{F}_t\right]}{E\left[\exp\left\{-r_i W_i^{S^*}(X;\mu^*,\sigma^*)\right\}\big|\,\mathcal{F}_t\right]} \\
&\quad + \frac{E\left[\exp\left\{-r_i W_i^{S^*}(X;\mu^*,\sigma^*)\right\}\big|\,\mathcal{F}_t\right]}{E\left[\exp\left\{-r_i W_i^{S^*}(X;\mu^*,\sigma^*)\right\}\big|\,\mathcal{F}_t\right]} \\
&= E\left[-\exp\left\{-r_i\left(W_i^{\mathbf{S}^R(t)}(X;\mu^*,\sigma^*)\right.\right.\right. \\
&\quad \left.\left.\left.- \; W_i^{S^*}(X;\mu^*,\sigma^*)\right)\right\}\Big|\,\mathcal{F}_t\right] + 1 \\
&= -\exp\left\{-r_i k_{it}^R\right\} + 1,
\end{aligned}$$

which implies $\exp\left\{r_i k_{it}^R\right\} \ge 1$, so $k_{it}^R \ge 0$, for all $i \in N$. Moreover, by efficiency $\sum_i k_{it}^R = 0$. Hence, $k_{it}^R = 0$ for all i and t and h^t, thus, $((\mathbf{S}_i^R)_i,\mu^*,\sigma^*) = ((\mathbf{S}_i^*)_i,\mu^*,\sigma^*)$; a contradiction.

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