ECON 415 – Game Theory
Homework 1: Strategic (Normal) Form Games

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Homework assignment is out of 100+10 (bonus) points. Randomly selected four questions and the bonus will be graded. You have to hand it in until February 12, Friday, at the beginning of the lecture.

1. Consider the following two player game:

	L	M	R
T	$2,-1$	$4,2$	$2,0$
C	$3,3$	$0,0$	$1,1$
B	$1,2$	$2,8$	$5,1$

   (a) What are the strategies that survive IESDS?

   Strategy R is strictly dominated by the mixture $\frac{1}{2}L + \frac{1}{2}M$. After R is eliminated, B is strictly dominated by T. The answer is $\{T, C\} \times \{L, M\}$.

   (b) At each step of the elimination what were the rationality assumptions?

   At the first step we assume that player 2 is rational, at the second step we assume player 1 is rational and knows that player 2 is rational.

   (c) Find all Nash equilibria including the mixed one.

   It is easy to see that (T, M) and (C, L) are pure strategy Nash equilibria. The mixed strategy Nash equilibrum is $(\frac{1}{2}T + \frac{1}{2}C, \frac{4}{5}L + \frac{1}{5}M$. Players choose the mixtures that make the opponent indifferent.

2. Discrete First-Price Auction: An item is up for an auction. There are two players. Player 1 values the item at 3 while player 2 values the item at 5. Each player can bid either 0, 1 or 2. If player i bids more that player j then player i wins the item and pays his bid while the loser does not pay. If both players bid the same amount, then a coin

is tossed to determine the winner, and the winner gets the item and pays his bid while the loser pays nothing.

(a) Write down this game as a normal-form game and in matrix form.

- Players: $N = \{1, 2\}$,
- Strategy set for each player $i \in N$: $S_i = \{0, 1, 2\}$,
- Payoff of each player:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i \quad \text{if} \quad b_i > b_j \\ \frac{1}{2}(v_i - b_i) \quad \text{if} \quad b_i = b_j \\ 0 \quad \text{if} \quad b_i < b_j \end{cases}$$

Table 1: Payoff Matrix

	0	1	2
0	1.5, 2.5	0,$\underline{4}$	0, 3
1	$\underline{2}$, 0	$\underline{1}$, 2	0, $\underline{3}$
2	1, 0	$\underline{1}$,0	$\underline{0.5}$, $\underline{1.5}$

(b) Does any player have strictly dominated strategy?

For player 2, bidding 0 is strictly dominated by bidding 2.

(c) Which strategies survive IESDS?

As 0 is strictly dominated by player 2, a rational player 2 will never play it. After bidding 0 is eliminated, in the reduced game, bidding 0 is strictly dominated by bidding 2 for player 1. So, an the second round of elimination, 0 can be deleted for player 1. In the smaller (2×2) game, bidding 1 is strictly dominated by bidding 2 for player 2. So, in the third round, we can eliminate 1 for player 2. Then, in the final round, one can eliminate bidding 1 for player 1. Hence, the only strategy profile that survives IESDS is $(2, 2)$.

(d) Find the set of pure Nash equilibria.

The unique NE is $(2, 2)$. Remember the theorem we discussed in the lecture: If IESDS gives a unique strategy profile, this must be the only NE of this game.

3. **Domination with mixed strategies** Consider the following two-person game.

	L	R
T	$4, 2$	$0, 0$
M	$0, 0$	$4, 2$
B	$1, 1$	$1, 1$

(a) Let p be the probability player 1 plays T and q be the probability that player 2 plays L. What is the range of values p can take so that the mixture of T and M strictly dominates B, i.e. find the set of mixed strategies that strictly dominates B?

As p is the probability player 1 plays T, $1-p$ is the probability player 1 plays M in the mixture of T and M. When player 2 plays L, this mixture gives a payoff of $4.p$. When player 2 plays R, the mixture gives a payoff of $4.(1-p)$. These payoffs should be strictly higher than what B pays off, which is 1. So, $4p > 1$ and $4(1-p) > 1$ gives a range of $\frac{1}{4} < p < \frac{3}{4}$.

(b) Given that B is strictly dominated by a mixture of T and M, find and *draw* the best responses of each player and the set of all (pure and mixed) NE.

As B is strictly dominated, it will not be played in any Nash equilibrium. The game becomes a Battle of the Sexes game.

	L	R
T	$4, 2$	$0, 0$
M	$0, 0$	$4, 2$

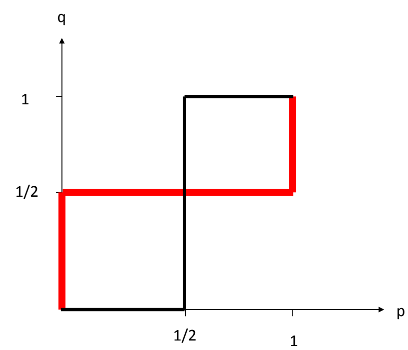
Let p be the probability that player 1 plays T and q be the probability that player 2 plays L. The best responses of each player can be written as follows:

$$p \equiv BR_1(q) = \begin{cases} 1, \quad \text{if} \quad q \geq \frac{1}{2} \\ [0,1] \quad \text{if} \quad q = \frac{1}{2} \\ 0 \quad \text{if} \quad q \leq \frac{1}{2} \end{cases}$$

Similarly,

$$q \equiv BR_2(p) = \begin{cases} 1, \quad \text{if} \quad p \geq \frac{1}{2} \\ [0,1] \quad \text{if} \quad p = \frac{1}{2} \\ 0 \quad \text{if} \quad p \leq \frac{1}{2} \end{cases}$$

So, the set of NE is $\{(p,q) : (0,0),(1,1),(\frac{1}{2},\frac{1}{2})\}$ as it is the intersection of best responses, which can be drawn as below:



4. Tragedy of commons (public good problem): Suppose that there are two firms each choosing how much to produce *simultaneously*. Each production consumes some of the clean air. There is a total amount of clean air that is equal to K and the consumption of clean air comes out of this common resource. Each player i (firm) chooses its own consumption of clean air for production, which is denoted by $k_i \geq 0$. The amount of clean air left is $K - \sum_{j=1}^{2} k_j$. The firm enjoys not only the consumption of the clean air for its production but also the clean air left after the production. Thus, its payoff function is given as:

$$u_i(k_i, k_{-i}) = \ln(k_i) + \ln(K - \sum_{j=1}^{2} k_j)$$

Answer the questions below for this environment.

(a) Describe this situation as a strategic game.

i. The set of players $N = \{1, 2\}$,

ii. The set of actions for each $i \in N$, $k_i \in A_i = [0, \infty\}$,

iii. The payoff function for each $i \in N$, $u_i(k_i, k_{-i}) = \ln(k_i) + \ln(K - \sum_{j=1}^{2} k_j)$.

(b) Compute and draw the best response correspondence for each firm. Then find the NE.

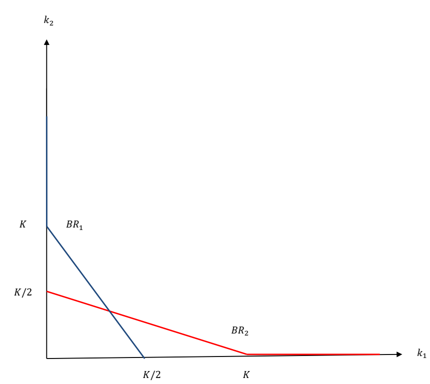
The best response correspondence –function (why?)– of firm $i \in N$ can be found by the FONC of its optimization problem:

$$\max_{k_i \in A_i} \ln(k_i) + \ln(K - \sum_{j=1}^{2} k_j).$$

FONC gives $k_i^* = \frac{K-k_j}{2}$. As $k_i^* \geq 0$, the best response can be written as

$$BR_i(k_j) = \begin{cases} \frac{K-k_j}{2}, & \text{if} \quad K \geq k_j \\ 0, & \text{otherwise.} \end{cases}$$

Since this is a symmetric game, it must be so that $k_1^* = k_2^* = k^*$ in equilibrium. Hence, the NE outcome is $(k_1^*, k_2^*) = (\frac{K}{3}, \frac{K}{3})$.



(c) Is the Nash equilibrium outcome Pareto efficient? If not, give an example of an efficient strategy profile.

NO. We can find another strategy profile which makes both players strictly better off (Pareto improvement). For instance, $(k_1, k_2) = (\frac{K}{4}, \frac{K}{4})$. To find the set of efficient profiles, we can maximize the weighted sum of society's payoffs. Since this is a symmetric game, we maximize,

$$\max_{k_1,k_2} \sum_{i\in N} \big(\ln(k_i) + \ln(K - \sum_{j=1}^{2} k_j)\big).$$

FONCs with respect to k_1 and k_2 result in,

$$\begin{aligned} \frac{1}{k_1} &= \frac{2}{K - k_1 - k_2} \\ \frac{1}{k_2} &= \frac{2}{K - k_1 - k_2} \end{aligned}$$

As $k_1 = k_2 = k$, one gets $k = \frac{K}{4}$.

(d) Which actions survive one round of iterated elimination of strictly dominated actions? What is the rationality requirement for one round of iteration? Justify your answer.

Note that the best response of firm i implies that it increases as k_j decreases. The maximum optimal amount that would be chosen by firm i even when firm j chooses the minimum amount $k_j = 0$ is $\frac{K}{2}$. Thus, it can be easily verified that any amount $k_i > \frac{K}{2}$ is strictly dominated by $\frac{K}{2}$. The players being rational is the only requirement for the elimination of the strictly dominated actions as a strictly dominated action can never be a best response against any strategy that the opponent could have chosen.

(e) Which strategy profiles survive IESDS? Is this game dominance solvable? What is the rationality requirement (rationality, k-level knowledge, common knowledge)? Justify your answer.

After the strictly dominated actions of both players are eliminated in the first round of iteration, we are left with a smaller game where the set of actions are $k_i, k_j \in [0, \frac{K}{2}]$. Then the best response of player i implies that $k_i = \frac{K-k_j}{2} \geq \frac{K}{4}$ as $k_j \leq \frac{K}{2}$. Thus, any profile that is lower than $\frac{K}{4}$ is eliminated in the second round and the truncated game we get has $[\frac{K}{4}, \frac{K}{2}]$ as the action set. If we continue this process, $[k_{min}, k_{max}]$ satisfies:

$$\begin{aligned} k_{min} &= \frac{K - k_{max}}{2} \\ k_{max} &= \frac{K - k_{min}}{2} \end{aligned}$$

which implies $k_{min} = k_{max} = \frac{K}{3}$ at infinitum. Infinitely many iterations require common knowledge of rationality.

5. **Bertrand competition with homogenous products:** Suppose that there are two firms with unit costs $c > 0$. They choose prices for the same product they produce simultaneously. The one with the lower price captures the entire market. In case of a tie, they share the market equally. The total market demand is equal to 1.

   (a) Write down the strategic form of this game.

   - Players: $N = \{1, 2\}$,
   - Strategies: $p_i \in S_i = [0, \infty)$,
   - Payoffs:

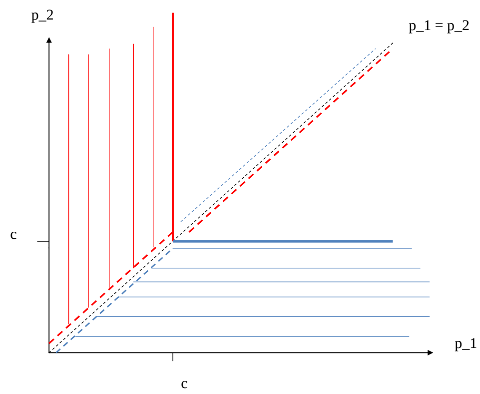
$$u_i(p_i, p_j) = \begin{cases} p_i - c \quad \text{if} \quad p_i < p_j \\ \frac{1}{2}(p_i - c) \quad \text{if} \quad p_i = p_j \\ 0 \quad \text{if} \quad p_i > p_j \end{cases}$$

   (b) Compute and draw the best response correspondences. Find Nash equilibria.

   The best response of firm i depends on p_j and c.

$$BR_i(p_j) = \begin{cases} (p_j, \infty) \quad \text{if} \quad p_j < c \\ [p_j, \infty) \quad \text{if} \quad p_j = c \\ p_j - \epsilon \quad \text{if} \quad p_j > c \end{cases}$$

   The red graph is BR_2 and the blue one is BR_1. The only intersection point is $(p_1, p_2) = (c, c)$ which is the unique Nash equilibrium.



6. State whether the following statements are true or false. Prove if it is true and give a counterexample if it is false.

   (a) A strictly dominated action profile cannot be Nash equilibrium.

   TRUE.

   Proof: Suppose that $a^* = (a_1^*, ..., a_n^*)$ is a strictly dominated action profile. Then, for some player $i \in N$, a_i^* is strictly dominated by some $b_i \neq a_i \in A_i$ i.e.

$$u_i(b_i, a_{-i}) > u_i(a_i^*, a_{-i}) \quad \text{for all} \quad a_{-i} \in A_{-i}.$$

   Since, this is true for all $a_{-i} \in A_{-i}$, it must be true in particular for a_{-i}^*. Hence,

$$u_i(b_i, a_{-i}^*) > u_i(a_i^*, a_{-i}^*) \quad \text{for all} \quad a_{-i} \in A_{-i}$$

   which implies that $u_i(a_i^*, a_{-i}^*) \not\geq u_i(a_i, a_{-i}^*)$ for all $a_i \in A_i$ (for instance, there is $b_i \in A_i$ that this does not hold). Thus, $a_i^* \notin BR_i(a_{-i}^*)$ implying it is not part of a Nash equilibrium profile. This completes the proof.

   (b) A Nash equilibrium profile cannot involve a play of weakly dominated action.

   FALSE.

	L	R
T	$2,2$	$0,0$
B	$0,0$	$0,0$

(c) Every finite normal form game has a pure strategy Nash equilibrium.

FALSE. Example: Matching Pennies

(d) Every finite normal form game has a completely mixed (not pure) strategy Nash equilibrium.

FALSE. Prisoners' dilemma (Any game where there is SDE, there is unique Nash equilibrium that is in pure strategies.)

7. **BONUS: (War of Attrition)** Consider a situation where there are two parties disputing over an object. Assume that the value party i attaches to the object is $v_i > 0$. Let time be a continuous variable that starts from 0 and runs indefinitely. Each unit of time that passes before the dispute is settled (i.e. one of the parties concedes) costs each party one unit of payoff. Thus, if player i concedes first, at time t_i, her payoff is $-t_i$ (she spends t_i units of time and doesn't obtain the object.) If the other player concedes first, at time t_j, player i's payoff is $v_i - t_j$ (she obtains the object after t_j units of time). If both players concede at the same time, player i's payoff is $\frac{1}{2}v_i - t_i$ where t_i is the common concession time.

(a) Write down the strategic form of this game.

- Players: $N = \{1, 2\}$,
- Strategies: $t_i \in S_i = [0, \infty)$,
- Payoffs:

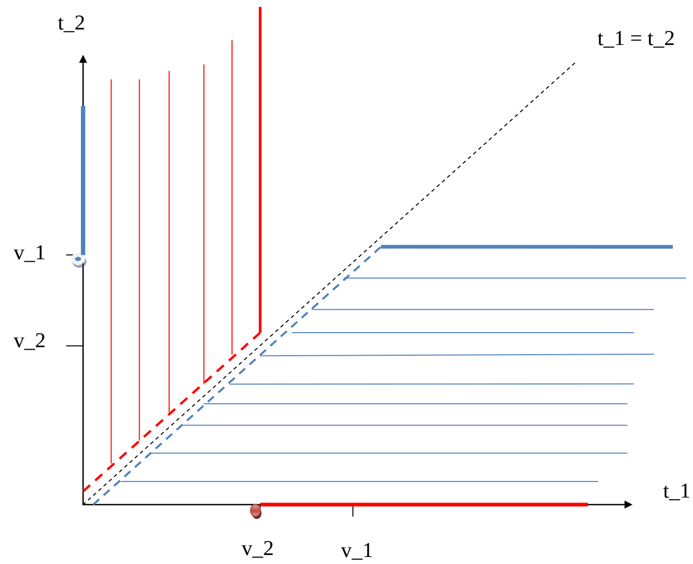
$$u_i(t_i, t_j) = \begin{cases} -t_i \quad \text{if} \quad t_i < t_j \\ \frac{1}{2}v_i - t_i \quad \text{if} \quad t_i = t_j \\ v_i - t_j \quad \text{if} \quad t_i > t_j \end{cases}$$

(b) Derive and draw the best response correspondences and find the set of Nash equilibria. (You can assume $v_1 > v_2$)

The best response of each player i depends on what her opponent chooses (t_j) as well as her value v_i.

$$BR_i(t_j) = \begin{cases} (t_j, \infty) \quad \text{if} \quad t_j < v_i \\ \{0\} \cup (t_j, \infty) \quad \text{if} \quad t_j = v_i \\ \{0\} \quad \text{if} \quad t_j > v_i \end{cases}$$

The blue graph is BR_1 and the red one is BR_2.



Hence, the set of Nash equilibria of this game $NE = \{(t_1, t_2) : (t_1 = 0 \quad \text{and} \quad t_2 \geq v_1) \cup (t_2 = 0 \quad \text{and} \quad t_1 \geq v_2)\}$.